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LETTER TO THE EDITOR

Cross correlations in nonlinear dynamics

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Abstract

It is commonly assumed that in statistical closures describing fluid or magnetohydrodynamic turbulence, the only ingredient needed for an understanding of the turbulent evolution is the energy spectrum. This assumption is shown to be valid only in very special cases. The key feature invalidating the unique significance of the energy spectrum is the existence of more than one non-vanishing quadratic invariant for a non-dissipative system, such as the presence of non-zero energy and helicity invariants. The latter plays a key role in the evolution of a turbulent MHD dynamo. These results have serious implications for the development of practical closures for inhomogeneous turbulence.

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Formal theoretical calculations of the behaviour of turbulent fluids focus on the evolution of the average value of elements of the energy spectrum. For example, if $\mathbf{v}(\mathbf{r}, t) = \sum_{\mathbf{k}} X_{\mathbf{k}}(t)\vec{\xi}_{\mathbf{k}}(\mathbf{r})$ represents a spectral expansion of the velocity field in some set of orthonormal basis functions, $\{\vec{\xi}_{\mathbf{k}}(\mathbf{r})\}$, with an associated set of spectral coefficients, $\{X_{\mathbf{k}}(t)\}$, labelled by three 'quantum' numbers symbolized by $\mathbf{k}$, then the quantities on which turbulence closures have been focused are $\{\langle|X_{\mathbf{k}}(t)|^2\rangle\}$. For the case of homogeneous turbulence, translational invariance sanctions this focus when the basis functions consist of plane waves; i.e., in this case $\mathbf{k}$ actually represents a wave vector and $\vec{\xi}_{\mathbf{k}}(\mathbf{r}) \propto \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})$. However, we have also found the focus on ensemble-averaged spectral components of the energy to be useful in the description of inhomogeneous turbulence [1]. Here the basis functions no longer consist of plane wave by virtue of the geometric constraints of inhomogeneity. In this case, we termed our intuitively motivated *a priori* justification for focusing on only the elements of the energy spectrum a 'random-phase approximation' (RPA) and demonstrated its utility by comparison with direct numerical simulations [2].

In this paper, however, we shall demonstrate the limitations of validity of the RPA. To do so, we have designed the following dynamical system of equations which incorporate the salient features of a truncated set of spectral equations for the Euler equation for an

incompressible homogeneous fluid,

$$\begin{aligned}
&\frac{\mathrm{d}X_i(t)}{\mathrm{d}t} = \sum_{j,k=-N}^{N} c_{ijk} X_j(t) X_k(t) \\
&c_{ijk} = \mathrm{sgn}(ijk)\epsilon_{ijk}(s_j|j| - s_k|k|)(s_i|i| + s_j|j| + s_k|k|)A(i,j,k) \qquad (1)\\
&s_l \equiv \pm 1
\end{aligned}$$

in which ϵ_{ijk} is a totally antisymmetric array defined to be equal to +1 when the subscripts are in monotonically increasing order or cyclical permutations thereof, equal to 0 when any two subscripts are equal, and equal to -1 otherwise. All quantities are assumed to be real. The subscript i takes on integer values from $-N$ to N. If $A(i, j, k)$ is chosen to be a totally symmetric function of its arguments and if for each allowed value of l, s_l is independently fixed at either +1 or -1, then one can verify readily that W is constant, where

$$W \equiv \sum_{k=-N}^{N} X_k^2(t). \qquad (2)$$

For our numerical calculations, we have chosen the following definition for $A(i, j, k)$:

$$\begin{aligned}
A(i,j,k) \equiv &\left[\frac{(|i|+|j|+|k|)(|i|+|j|-|k|+1)(|j|+|k|-|i|+1)(|k|+|i|-|j|+1)}{16|i||j||k|}\right]^{\frac{1}{2}} \\
&\text{for} \quad ijk \neq 0, \quad |i|+|j| \geqslant |k|, \quad |j|+|k| \geqslant |i|, \quad |k|+|i| \geqslant |j| \qquad (3)\\
&A(i,j,k) \equiv 0 \qquad \text{otherwise.}
\end{aligned}$$

The structure of this toy model is designed to mimic closely the structure of the three-dimensional Euler fluid in which the basis functions are eigenfunctions of the curl operator, the Chandrasekhar–Kendall functions [3]. (In the context of Navier–Stokes literature, in addition to [1], this choice of basis function has an illustrious history [4].) The scalar indices taking on values on the domain $-N, N$ are the analogue of wave vector indices.

We compare two cases: case A in which $W = 1.116$ and case B in which $W = 3.00$. In both cases, $N = 10$ and $X_0(t) = 0\ \forall t$. In case A, we set $s_i = \mathrm{sgn}(i)\ \forall i$ and we set $X_i(0) = \frac{1}{|i|+1}$ for $i \neq 0$. In case B, we set $s_i = +1\ \forall i$ and we set $X_i(0) = 1$, for $i = 8, 9, 10$ as the only non-vanishing components of $\mathbf{X}$. For case A, W varied by no more than $7.2 \times 10^{-3}\%$ of its initial value during the period, $t = 0$ to $t = 180$, of the numerical computation. For case B, W varied by no more than $1.9 \times 10^{-2}\%$ of its initial value during the computation period, $t = 0$ to $t = 100$.

As one might expect, a mode's evolution appears to be rather random. In figure 1, which exhibits spectral evolution only over the last unit of time, one observes no meaningful difference between the outputs, $X_{-8}(t)$ of case A and $X_{-5}(t)$ of case B. Qualitatively, the output seems insensitive to the different initial conditions.

Now suppose we consider an arbitrary N-dimensional, time-independent, orthogonal transformation that transforms the X to $\tilde{X}$,

$$X_j(t) = \sum_{i=-N}^{N} O_{ji} \tilde{X}_i(t) \qquad \mathbf{O}^T\mathbf{O} = \mathbf{I} \qquad (4)$$

and insert this expression into the evolution equation for the X, equation (1). We would find an evolution equation for the new $\tilde{X}$ having a similar structure; namely,

$$\frac{\mathrm{d}\tilde{X}_i(t)}{\mathrm{d}t} = \sum_{j,k=-N}^{N} \tilde{c}_{ijk} \tilde{X}_j(t) \tilde{X}_k(t). \qquad (5)$$

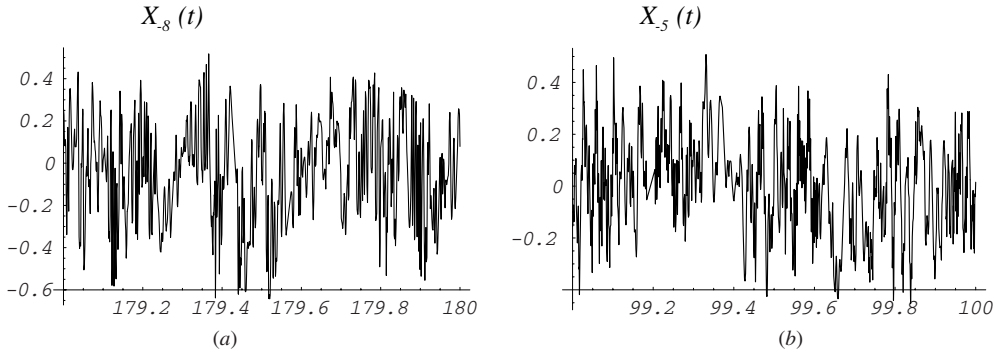

Figure 1. Examples of the evolution of spectral coefficients during the last unit of time computed. (*a*) A typical coefficient for case A: $K = O(10^{-5})$, $W = 1.116$; (*b*) a typical coefficient for case B: $K = 27$, $W = 3$.

Of course, if we were to look at typical traces of the evolution of elements of the coefficients, $\{\tilde{X}_i\}$, they would look qualitatively similar to those traces depicted in figure 1. One's first inclination would be to anticipate that only diagonal elements of $\langle \mathbf{X}(t)\mathbf{X}(t)\rangle$ and $\langle \tilde{\mathbf{X}}(t)\tilde{\mathbf{X}}(t)\rangle$ would be non-vanishing, namely $\langle X_i(t)X_i(t)\rangle$ and $\langle \tilde{X}_i(t)\tilde{X}_i(t)\rangle$, $i = -N, -N+1, \ldots, -1, 1, \ldots, N-1, N$. Such an expectation arises because of the apparent lack of any constraint that is operant in the case of the nondiagonal values and because of the highly nonlinear nature of the evolution equation that 'stirs' these coefficients. We have termed the implementation of this intuitive assumption a 'random-phase approximation' (RPA) [1].

To check the validity of the RPA, we define the expectation value of a quantity $Q(t)$, $\langle Q(t)\rangle$ by

$$\langle Q(t)\rangle = \frac{\sum_{i=1}^{250} Q(t_i)}{250} \tag{6}$$

in which the set of times, $\{t_i\}$, are 250 randomly selected points during the last unit of time of the computations. For t within this last unit, we then define the fluctuation quantity,

$$\delta Q(t) = Q(t) - \langle Q(t)\rangle \tag{7}$$

which obviously satisfies $\langle \delta Q(t)\rangle = 0$. It follows that

$$W = \bar{W} + \delta W \tag{8}$$

in which

$$\bar{W} \equiv \sum_{i=-N}^{N} \langle X_i(t)\rangle^2 \tag{9}$$

$$\delta W \equiv \sum_{i=-N}^{N} \left\langle \delta X_i^2(t)\right\rangle. \tag{10}$$

$\bar{W}$ can be thought of as the energy in the mean field and δW as the energy in the fluctuations.

We can introduce a normalized correlation matrix, $\mathbf{M}(\mathbf{X})$, by

$$M_{ij}(\mathbf{X}) \equiv \frac{\langle X_i(t)X_j(t)\rangle}{\left(\left\langle X_i^2(t)\right\rangle\left\langle X_j^2(t)\right\rangle\right)^{\frac{1}{2}}}. \tag{11}$$

Next we define a mean magnitude of a nondiagonal correlation by

$$C(\mathbf{X}) \equiv \frac{1}{N(2N-1)} \sum_{i'<j'=-N}^{N}{}' M_{i'j'}(\mathbf{X}) \tag{12}$$

where we use the prime on the summation index to indicate that we are excluding a summation on the value of the index equal to zero.

We list our results for cases A and B:

	case A	case B
W	1.116	3.00
$\bar{W}$	0.063	0.082
δW	1.053	2.918
$C(\mathbf{X})$	0.107	0.127
$C(\delta\mathbf{X})$	0.108	0.122
$C(\tilde{\mathbf{X}})$	0.105	0.355
$C(\delta\tilde{\mathbf{X}})$	0.105	0.366

We obtained the tilde variables by performing a randomly chosen orthogonal transformation on the original variables.

If we think of W as energy and $\mathbf{X}$ as spectral coefficients of a flow, then for both cases A and B we note that there is little mean energy, most of the energy being in the fluctuations. Thus it is not surprising that for each case, $C(\mathbf{X}) \approx C(\delta\mathbf{X})$ and $C(\tilde{\mathbf{X}}) \approx C(\delta\tilde{\mathbf{X}})$. Note that the diagonal elements of the correlation matrix, $\mathbf{M}(\mathbf{X})$, are unity. Although we can expect some sampling-error-induced imprecision in the values of the C [5], what demands explanation is the discrepancy between the values of $C(\delta\tilde{\mathbf{X}})$ and $C(\tilde{\mathbf{X}})$ with those of $C(\delta\mathbf{X})$ and $C(\mathbf{X})$, respectively, in case B. There is no such discrepancy in case A. Clearly, the random-phase approximation is not generally valid when applied to case B.

The discrepancy signals the presence of additional constraints on our ensemble of values. Indeed equation (1), and consequently equation (5), has a second invariant quantity, which we shall denote by K. This invariant, defined by

$$K \equiv \sum_{k=-N}^{N} s_k |k| X_k^2(t) \tag{13}$$

is analogous to the kinetic helicity invariant of a Euler fluid under suitable boundary conditions. Over the 180 units of time in the numerical computation of case A, K varied no more than 8×10^{-5} starting from a value of 0. Over the 100 units of time in the computation of case B, K varied by no more than $1.5 \times 10^{-2}\%$ of its initial value, 27. If we express K using $\tilde{\mathbf{X}}$ variables, we find that it is given by the nondiagonal structure:

$$K = \sum_{i,j,k=-N}^{N} \tilde{X}_j O_{ji}^{-1} s_i |i| O_{ik} \tilde{X}_k. \tag{14}$$

Thus, in the $\tilde{\mathbf{X}}$ representation, neglect of the off-diagonal elements (not present in the original $\mathbf{X}$ representation) would be tantamount to destroying the invariance of K. Such terms cannot be set equal to zero. The presence of a second invariant of the dynamical system of equations leads us to a preferred choice of basis functions, namely the set in which the correlation matrix is diagonal. We shall refer to such a set as the *principal* set. In the general physical case, the quantities corresponding to W and K, analogous to energy and helicity, are always real so that one can always perform a unitary transformation to obtain such a basis.

In analogy with equations (9) and (10), we can define $\bar{K}$ and δK:

$$\bar{K} \equiv \sum_{k=-N}^{N} s_k |k| \langle X_k(t) \rangle^2 \tag{15}$$

$$\delta K \equiv \sum_{k=-N}^{N} s_k |k| \langle \delta X_k^2(t) \rangle. \tag{16}$$

For case A, we find that $K = 7 \times 10^{-5}$, with $\bar{K} = -0.099\,13$ and $\delta K = 0.099\,20$. For case B, we find that $K = 27.00$, with $\bar{K} = 0.57$ and $\delta K = 26.43$.

We can use the conservation of K to predict the correlation matrix for our dissipationless case. As is well known, the presence of the second invariant, K, affects the values of the fluctuation spectrum in the basis in which K is diagonal. To see this, we compare the values of the fluctuation energy spectra in the diagonal basis for cases A and B, against what would be expected from the absolute equilibrium ensemble density of states distribution [6],

$$D_{eq}(\delta \mathbf{X}) = \frac{\exp[-\beta \delta W(\delta \mathbf{X}) - \alpha \delta K(\delta \mathbf{X})]}{\left[\prod_{i'=-N}^{N} \int_{-\infty}^{\infty} \mathrm{d}(\delta X_{i'})\right] \exp[-\beta \delta W(\delta \mathbf{X}) - \alpha \delta K(\delta \mathbf{X})]} \tag{17}$$

in which $\delta W(\mathbf{X})$ and $\delta K(\mathbf{X})$ are the expressions on the right-hand side of equations (10) and (16), respectively. According to the absolute equilibrium ensemble, the average value of δX_k^2, $\langle \delta X_k^2 \rangle_{eq}$, is given by

$$\begin{aligned} &\left[\prod_{i'=-N}^{N} \int_{-\infty}^{\infty} \mathrm{d}(\delta X_{i'})\right] D_{eq}(\delta \mathbf{X}) \delta X_k^2 \\ &\quad = -\frac{\partial}{\partial \beta} \log \left[\int_{-\infty}^{\infty} \mathrm{d}(\delta X_k) \delta X_k^2 \exp\left(-\beta \delta X_k^2 - \alpha s_k |k| \delta X_k^2\right)\right] \\ &\quad = \frac{1}{2(\alpha s_k |k| + \beta)}. \end{aligned} \tag{18}$$

Using equation (4), we observe that the average value of a general correlation in a nondiagonal representation can be obtained using the absolute equilibrium ensemble:

$$\langle \delta \tilde{X}_i(t) \delta \tilde{X}_j(t) \rangle_{eq} = \sum_{k=-N}^{N} O_{ik}^T \langle \delta X_k(t) \delta X_k(t) \rangle_{eq} O_{kj}. \tag{19}$$

Note that only for a case of energy equipartition, a case in which $K = 0$, do the values of $\langle \delta \tilde{X}_i(t) \delta \tilde{X}_j(t) \rangle_{eq}$ for $i \neq j$ necessarily vanish.

The absolute equilibrium-ensemble-averaged values of δW and δK follow

$$\delta W_{eq} = \sum_{j'=-N}^{N} \frac{1}{2(\alpha s_{j'} |j'| + \beta)} \tag{20}$$

$$\delta K_{eq} = \sum_{j'=-N}^{N} \frac{s_{j'} |j'|}{2(\alpha s_{j'} |j'| + \beta)}. \tag{21}$$

Observe that in case A, in which δK was observed to be negligible, we had chosen $s_i = \mathrm{sgn}(i)$. For this choice, we note that when we set $\alpha = 0$, equations (20) and (21) yield an equipartition spectrum of 0.0527 for the components of δW_{eq} and 0 for the components of

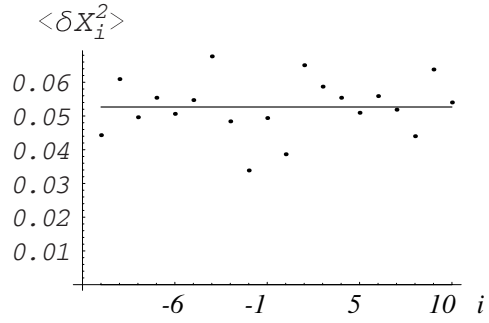

Figure 2. Case A, the case of expected energy equipartition: the dots depict the values of the 20 non-zero components of the fluctuation energy spectrum, δW. Their mean value is indicated by the solid line.

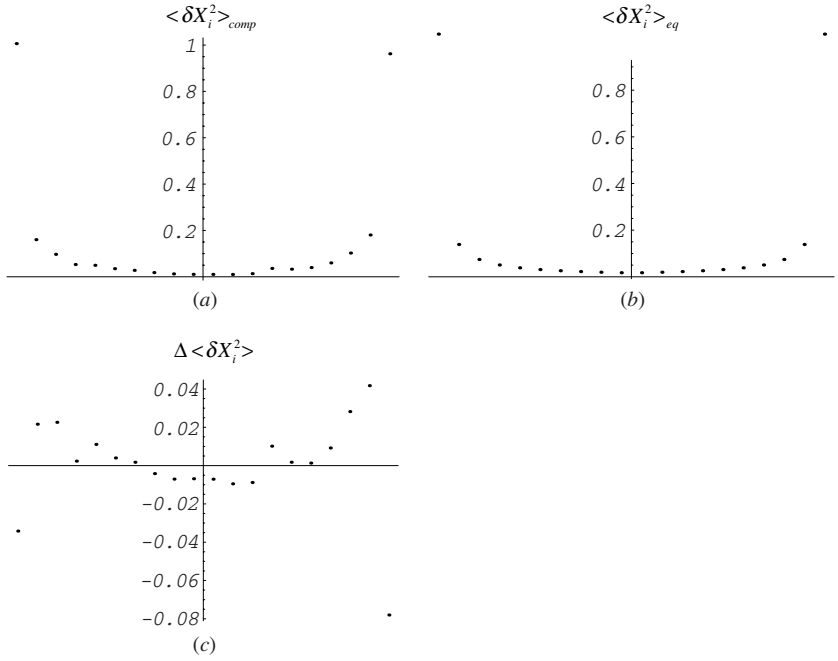

Figure 3. Case B, the case with non-zero K. The modal indices of the non-zero components of the fluctuation energy spectrum δW increase from -10 to 10 as one proceeds from left to right: (*a*) the dots depict their computationally evolved values; (*b*) the dots depict their values obtained using the absolute equilibrium ensemble; (*c*) the dots depict the discrepancy: the values of (*a*) minus the values of (*b*).

δK_{eq}. For case B, we insert the values of δW, 2.918, and δK, 26.43, into equations (20) and (21) and solve to find $\alpha = -2.855$, $\beta = 29.025$.

In figure 2, we exhibit the fluctuation energy spectrum for case A during the last unit of time computed. The solid line runs through at the level of the fluctuation spectrum given by the absolute equilibrium ensemble. In figure 3(*a*), we exhibit the evolved fluctuation energy spectrum for case B during the last unit of time computed; in figure 3(*b*), we exhibit the fluctuation spectrum given by the absolute equilibrium for case B; in figure 3(*c*), we depict

the difference between the results of figures 3(a) and (b). In both figures 2 and 3, the $i = 0$ mode is excluded.

These results have significant implications for the development of practical closures for inhomogeneous turbulence. Essentially all formal theoretical closures provide a set of equations that is restricted to determining the evolution of the energy spectrum. Here we have demonstrated in a non-dissipative simplified model that such a restriction ignores crucial dynamical properties unless the basis utilized provides a representation in which the statistically averaged values of the cross correlations vanish. Although any complete set of orthonormal basis functions may be utilized when only the energy invariant of the dynamical system of equations exists, the presence of one or more additional quadratic invariants generally requires a unitary transformation to a principal set. Thus the presence of non-vanishing cross correlations contains information about the nature of the dynamics.

For the non-dissipative case with an energy invariant and additional quadratic invariants, $\{I_j\}$ (see for example, [7]), one would have an absolute equilibrium ensemble of the form $D_{eq} \propto \exp[-\beta W - \sum_j \alpha_j I_j]$. The diagonal representation would be found by first solving for the values of the Lagrange multipliers, β and $\{\alpha_j\}$ using the values of the quadratic invariants, and then finding the unitary transformation that diagonalizes $\sum_j \alpha_j I_j$.

An example of a calculation in which an infinite set of basis functions was chosen in such a way that the helicity invariant of magnetohydrodynamics, the magnetic helicity ($\int \mathbf{A} \cdot \mathbf{B}\, \mathrm{dv}$), was not diagonal can be found in [8]. The principal set of basis functions that diagonalizes the magnetic helicity was also obtained in that paper. The principal set turned out to be eigenfunctions of the curl operator, i.e., the Chandrasekhar–Kendall basis [3]. An analogous calculation using the Chandrasekhar–Kendall basis set was carried out by Montgomery *et al* [9]. The Chandrasekhar–Kendall functions also provide a diagonal representation for the kinetic helicity invariant ($\int \mathbf{v} \cdot \vec{\omega}\, \mathrm{dv}$) of the Euler equation for an ideal fluid.

For the case of a steady, driven-dissipative dynamical system, we would anticipate the continued presence of non-vanishing cross correlations in systems having more than one ideal quadratic invariant. However, in addition to their being affected by geometry and boundary conditions, these correlations now also may be affected by the properties of the driver and/or the dissipation. Even if the ideal system were to have only an energy invariant, these properties may introduce non-vanishing cross correlations into the corresponding driven-dissipative system. In a freely decaying dissipative turbulence, the principal set of basis functions might vary with time. Thus non-vanishing cross correlations can provide a rich source of useful information in the investigation of turbulent physical systems.

To summarize, we have created a simple dynamical system with two quadratic invariants. Utilizing this system, we have shown that the values of correlations between different amplitudes depend critically on the initial conditions, namely on the values of the system's quadratic invariants. These results augment the work of Shebalin [10], which emphasized the restrictions imposed by initial conditions in the presence of various discrete symmetries for Navier–Stokes and magnetohydrodynamic fluids.

The most significant consequence of our analysis is evident when applied to closures describing non-ideal inhomogeneous turbulence. With the exception of [1], there has been scant application of formal closures, such as the direct-interaction approximation, test-field model, and eddy-damped quasinormal Markovian closure to such systems. To permit efficient numerical calculations, such closures generally are applied to situations in which correlations between different spectral coefficients are zero. But we have just demonstrated that for inhomogeneous turbulence having more than one non-vanishing ideal quadratic invariant, these correlations would generally be zero only in a judiciously chosen representation, a representation that would not necessarily be known *a priori* and which may indeed evolve

with the turbulence. When an ideal invariant is nondiagonal, even the concept of its cascade, direct or inverse, through a spectrum loses meaning. In such cases, the turbulent physics information encapsulated in correlations may be able to be extricated efficiently only through the use of direct numerical simulations.

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